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AN EXTENSION OF BURNSIDE'S THEOREM TO INFINITE-DIMENSIONAL SPACES

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ABSTRACT

The classical Burnside's Theorem guarantees in a finite dimensional space the existence of invariant subspaces for a proper subalgebra of the matrix algebra. In this paper we give an extension of Burnside's Theorem for a general Banach space, which also gives new results on invariant subspaces.

Introduction

The fundamental Burnside Theorem for finite-dimensional representations says that an algebra of linear operators on a finite-dimensional space F, without invariant subspace, must be the algebra $L(F)$ of all linear operators on the space. In other words, if an algebra R is strictly contained in $L(F)$, then there exists $x \in F$ and $\varphi \in F^*$ s.t. for every $A \in R, (Ax, \varphi) = 0$.

In this paper we will extend this result to Banach spaces in the following way. In Theorem 1 below, let B be a Banach space, $L(B)$ the bounded linear operators on B and $K(B)$ the compact operators on B. Let $|||A|||$ denote the essential norm of A, i.e. the distance from A to the space of compact operators. A weakly closed algebra is an algebra closed in weak operator topology.

THEOREM 1: *Let R be a weakly closed subalgebra of $L(B)$, $R \neq L(B)$. Then there exists $x \in B^{**}$ and $y \in B^*$, $x \neq 0$ and $y \neq 0$, s.t. for every $A \in R$*

$$(1) \qquad |(x, A^*y)| \leq |||A|||.$$

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If B is finite-dimensional then $|||A||| = 0$, so Theorem 1 gives Burnside's theorem.

For a given subset $S \subset L(B)$ and a vector $x \in B$ put $Sx = \bigcup_{A \in S} Ax$ and $S' = \bigcup A^*$. We say that S is transitive if it doesn't have a nontrivial invariant subspace and S is essentially transitive if the conclusion of Theorem 1 is false for S. We say that a subalgebra $R \subset L(B)$ has the Pearcy–Salinas (PS) property if there exists net $\{A_\alpha\} \subset R$ and a nonzero operator $A \in L(B)$ such that:

$$\lim_\alpha (x, A_\alpha^* y) = (x, A^* y) \tag{2}$$

for every vector $x \in B^{**}$ and every functional $y \in B^*$ and

$$\lim_\alpha |||A_\alpha||| = 0. \tag{3}$$

Of course, every bounded operator is a weak limit of finite-dimensional ones, so it is the assumption $\{A_\alpha\} \subset R$ that makes the condition nontrivial.

Corollary 1: *Let R be a weakly closed proper subalgebra of $L(B)$ with PS property. Then the algebra R' is nontransitive.*

Proof: Let x and y be as in Theorem 1. Then for every pair of operators T_1, T_2 in R we have

$$\begin{aligned}|(x,\, T_2^* A^* T_1^* y)| &= \lim_\alpha |(x,\, T_2^* A_\alpha T_1^* y)| \\ &\le \overline{\lim_\alpha} |||T_2 A_\alpha T_1||| \\ &\le \overline{\lim_\alpha} \|T_2\|\ \|T_1\|\ |||A_\alpha||| \\ &= 0.\end{aligned}$$

It is easy to see that one of the three subspaces

$$\bigcap_{T \in R} \ker(T^*), \overline{R'y}, \overline{\operatorname{span}_{T_1, T_2 \in R}(T_2^* A^* T_1^* y)}$$

is a nontrivial invariant subspace for the algebra R'.

If the algebra R contains a nonzero compact operator then the PS property is trivially true for R. So we get the results obtained by the author's earlier techniques [8]. ◻

COROLLARY 2: *If a weakly closed proper subalgebra R of $L(B)$ contains a nonzero compact operator, then the algebra R' is nontransitive.*

Of course the existence of a hyperinvariant subspace in B^* for operators that commute with a compact operator follows from Corollary 2. It should be mentioned that the following theorem by Pearcy and Salinas [6] is not a consequence of the author's earlier techniques [see 7].

THEOREM (PEARCY–SALINAS): *Let T be a bounded operator on Hilbert space. Assume that there is a sequence of rational functions (s_n) s.t. $s_n(T)$ converges weakly to a non-zero operator A and s.t.*

$$|||s_n(T)||| \to 0 \text{ as } n \to \infty.$$

Then T has a nontrivial invariant subspace.

However, from Corollary 1 we get the following strengthening of the Pearcy–Salinas Theorem as

COROLLARY 3: *Let T be a bounded operator on Banach space. Assume that there is a nonzero operator A and net $\{A_\alpha\}$ of operators that commute with T s.t. (2) and (3) hold. Then T^* has a nontrivial hyperinvariant subspace.*

The commutant of T is a proper subalgebra of $L(B)$, so Corollary 3 is an immediate consequence of Corollary 1.

COROLLARY 4: *Let $S_\phi \subset L(B)$ be defined by $S_\phi = \{A \in R, |||A||| \leq 1\}$. Then there exists a nonzero functional $y \in B^*$ s.t. the set $S'_\phi y$ is not dense in B^*.*

Proof: Let x and y be as in Theorem 1. Then $\sup_{|||A^*|||<1} |(x, A^*y)| \leq 1$, so $S'_\phi y$ is not dense in B. □

This corollary was pointed out to us by S.W. Brown, who has obtained it and similar results in the case of a commutative algebra.

In Hilbert space Theorem 1 has the following equivalent formulation.

THEOREM 2: *Let R be as in Theorem 1. Let B be Hilbert space, then there exist two bounded nets (x_α) in B and (y_α) in B s.t. $x_\alpha \xrightarrow{w} x \neq 0$, $y_\alpha \xrightarrow{w} y \neq 0$ and for every $A \in R$*

$$< Ax_\alpha, \, y_\alpha > \to 0.$$

Proof of Theorems 1 and 2: Let B be a Banach space, Q a compact Hausdorff space, R a subalgebra of $L(B)$, WQ the Banach space of weak*-continuous functions $Q \to B^*$ with sup-norm, SQ the subspace of WQ consisting of the strongly continuous functions, NR the algebra of norm-continuous functions $Q \to R'$. Let $C = C(Q)$ be the algebra of all complex-valued continuous functions on Q. Every function $f \in C(Q)$ is a linear operator on the space WQ. We let f^* denote its adjoint.

Definition: Let M be a subspace of WQ, invariant under the algebra C. We say that $\theta \in M^*$ is a point functional if there exists a point $q \in Q$ s.t.

$$f^*\theta = \overline{f(q)}\theta \tag{4}$$

for every function $f \in C$.

If E is a subspace in a Banach space let U_E denote the unit ball of E and $E^\perp$ denote the annihilator of E in the dual space. We say that a functional $\theta \in M^*$ is a $\delta(q,x)$ functional or simply δ-functional if there exists a point $q \in Q$ and an element $x \in B^{**}$ s.t. if $h \in M$ then

$$\theta(h) = (x, h(q)). \tag{5}$$

LEMMA 1: *Let $\theta \in M^*$ be a point functional and assume $M \supset SQ$. Then the restriction $\chi = \theta|_{SQ}$ is a δ-functional.*

Proof: It is known [3] that the functional χ has the form

$$\chi(h) = \int_Q (d\chi(q), h(q)) \tag{6}$$

where $d\chi$ is a regular Borel measure on Q with values in B^{**}. Let $q \in Q$ be as in (4). For an open set $V = Q/\{q\}$ and for a given $\epsilon > 0$ there exists a closed set $F \subset V$ s.t. $\|d\chi|_V - d\chi|_F\| < \epsilon$. Then there exists function $f \in C(Q)$ s.t. $f(q) = 0$, $f|_F \equiv 1$. If G is given Borel set and $G \subset F$, then

$$d\chi(G) = \int_G d\chi = \int_G \bar{f} d\chi = f^*\chi = \overline{f(q)}\chi = 0,$$

so $d\chi|_F = 0$. This gives for every $\epsilon > 0$, $\|d\chi|_V\| < \epsilon$, and so $d\chi|_V = 0$. ◻

Let H be a Hilbert space with orthogonal basis $\{e_n\}, LIM(\alpha_n)$ a Banach limit in the space l^∞, Q compactification of integer numbers by $\{\infty\}$. Then $\theta(h) = LIM(h(n), e_n)$ is an example of a point functional that is not δ-functional.

Let $Q \times U_B$ be the topological product of the compact Q and the ball U_B with norm topology. The functions $s \in WQ$ and $f \in C(Q)$ define complex-valued functions $L(s)$ and $L_1(f)$ on $Q \times U_B$ for the formulas $L(s)(q,x) = (x, s(q)), L_1(f)(q,x) = f(q)$.

LEMMA 2: *$L(s), L_1(f)$ are continuous functions on*

$$Q \times U_B.$$

Proof: Let $q_0 \in Q$ and $x_0 \in U_B$. Let V_{q_0}, V_{x_0} be open subsets defined by

$$V_{q_0} = \{q \in Q : |(x_0, s(q) - s(q_0))| < \epsilon/2\},$$
$$V_{x_0} = \{x \in U_B : \|x - x_0\| < \epsilon/2|s|\};$$

then for all $(q, x) \in V_{x_0} \times V_{q_0}$ we have

$$\begin{aligned} |(x, s(q)) - (x_0, s(q_0))| &\leq |((x - x_0, s(q))| + |(x_0, s(q) - s(q_0))| \\ &\leq \frac{\epsilon}{2|s|}|s| + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

so $L(s)$ is continuous. The continuity of $L_1(f)$ is obvious. U_B is a completely regular space, so there exists the Cheh compactification F of the space $Q \times U_B$. The functions $L_1(f)$ and $L(s)$ in Lemma 2 have continuous extensions to all of F. Thus $L(M), L(N), L_1(C(Q))$ are subspaces of $C(F)$, $L(M) \supset L(N)$, and $L(M)$ and $L(N)$ are invariant for $L_1(C(Q))$. Moreover, the operators $L : M \to C(F)$ and $L_1 : C(Q) \to C(F)$ are isometries. □

LEMMA 3: *Let $\varphi \in L(M)^*$, $f \in C(Q)$ and*

$$(L_1 f)^* \varphi = \gamma \varphi,$$

then $f^*(L^*\varphi) = \gamma(L^*\varphi)$.

Proof: For a given function $h \in WQ$ we have $(h, f^*(L^*\varphi)) = (fh, L^*\varphi) = (L(fh), \varphi) = (L_1(f)L(h), \varphi) = (L(h), L_1(f)^*\varphi) = (L(h), \gamma\varphi) = (h, \gamma L^*\varphi)$. □

LEMMA 4: *Let M, N be subspaces of WQ, invariant for $C(Q)$. Let $M \supset N$ and let $\theta \in M^*$ be an extreme point of $U_{N^\perp}$. Then θ is a point-functional.*

The proof of this lemma is similar to De Branges' famous proof of the Stone–Weierstrass Theorem [2].

Proof: Let $\hat{\theta}$ be a functional in $L(M)^*$ s.t. $L^*(\hat{\theta}) = \theta$. Then, obviously $1 = |\hat{\theta}| = |\theta|$. By the Hahn–Banach and Riesz Theorems there exists a measure $d\mu \in C(F)^*$ which is an extension of $\hat{\theta}$ to space $C(F)$. Let f be a function in $C(Q)$, s.t.

$$0 \le f \le 1 \tag{7}$$

and $\hat{f} = L_1(f), d\mu_1 = \hat{f} d\mu, d\mu_2 = (1-\hat{f})d\mu, m_1 = \|d\mu_1\|, m_2 = \|d\mu_2\|$. Then $\{d\mu_1, d\mu_2, d\mu\} \subset L(N)^\perp$ and $m_1 + m_2 = \int_F \hat{f}|d\mu| + \int_F (1-\hat{f})|d\mu| = \int_F |d\mu| = 1$. If $\theta_1 = d\mu_1|_{L(M)}, \theta_2 = d\mu_2|_{L(M)}$, then $|\theta_1| + |\theta_2| \ge |\theta_1 + \theta_2| = 1$ and $|\theta_1| + |\theta_2| \le m_1 + m_2 = 1$, so we have $|\theta_2| = m_2, |\theta_1| = m_1$. If $m_1 = 0$, then $(L_1 f)^* \hat{\theta} = 0$, so by Lemma 3, $f^*\theta = 0$. If $m_2 = 0$ then $(1-f)^*\theta = 0$. If $m_1 \ne 0$ and $m_2 \ne 0$ then we have

$$\hat{\theta} = m_1 \frac{\theta_1}{m_1} + m_2 \frac{\theta_2}{m_2}, \ \|\frac{\theta_1}{m_1}\| = 1, \ \|\frac{\theta_2}{m_2}\| = 1.$$

The functional $\hat{\theta}$ is an extreme point in the ball $U_{L(N)^\perp}$, so we get $\theta_1/m_1 = \hat{\theta}$ or, by Lemma 3, $f^*\theta/m_1 = \theta$. So for any function f with property (7) the functional θ is an eigenvector and the functions of this type obviously generate $C(Q)$. The corresponding eigenvalue is a multiplicative functional on $C(Q)$, so Gelfand's theorem now gives the lemma. □

LEMMA 5: *Let $T \in K(B)$ and $h \in WQ$. Then $T^*h \in SQ$.*

We omit the simple proof.

Let R be an algebra in $L(B)$ and let $h \in WQ$, $\|h\| = 1$. By applying the algebra NR to h we get subspaces

$$N = NR(h), \ M = \text{ span } (N, SQ).$$

Then $M \supset N$ and M and N are invariant for the algebra $C(Q)$. Let φ be a point functional, $\varphi \in M^*$. Then, by Lemma 1 , $\varphi|_{SQ}$ is a $\delta(q, x)$ functional.

LEMMA 6: *If functional $\varphi \in N^{\perp}$, then for every operator $A \in R$*

$$(8) \qquad |(x, A^*h(q))| \leq 2|\varphi| \, |||A||| \, |h|.$$

Proof: Let T be a compact operator s.t. $\|A - T\| \leq |||A||| + \epsilon$. Then we have

$$\begin{aligned} |(x, A^*h(q))| &= |\delta A^*h| = |\delta(A^*h) - \varphi(A^*h)| \\ &= |(\delta - \varphi)A^*h| \leq |(\delta - \varphi)(A^* - T^*)h| + |(\delta - \varphi)T^*h|. \end{aligned}$$

By Lemma 5, $T^*h \in SQ$, so by Lemma 1, $\delta(T^*h) - \varphi(T^*h) = 0$. It is clear that $|\delta| \leq |\varphi|$, so we get

$$|(x, A^*h(q))| \leq 2|\varphi| \, \|A - T\| \, |h| < 2|\varphi|(|||A||| + \epsilon)|h|.$$

Since ϵ is arbitrary,

$$|(x, A^*h(q))| \leq 2|\varphi| \, |||A||| \, |h|.$$

□

Let $\sigma_F(T)$ be the subset of the spectrum of an operator $T \in L(B)$ consisting of the isolated points of finite multiplicity.

LEMMA 7: *Let the subspace $L \subset B$ have finite codimension. Assume $\alpha > 0$ and*

$$(9) \qquad \sup_{x \in L} \frac{\|Tx\|}{\|x\|} < \alpha.$$

Assume $\gamma \in \sigma(T)$ and $|\gamma| > \alpha$. Then $\gamma \in \sigma_F(T)$

Proof: It is sufficient to prove this lemma in the case that γ is a boundary point of $\sigma(T)$. Let P be a bounded projection on L. Then the operator $T_1 = (T - \gamma I)P$ is semi-Fredholm.

It is clear that $\ker T_1 = \ker P$ and if $x \in L$, then

$$|T_1 x| = |(T - \gamma I)Px| = |(T - \gamma I)x| \geq (|\gamma| - \alpha)|x|.$$

So T_1 has finite-dimensional kernel and closed range. $T - \gamma I$ is a finite-dimensional perturbation of T_1, so by [1, Corollary 1.3.7] $T - \gamma I$ is also semi-Fredholm. Moreover, $T - \gamma I$ is a limit of invertible operators so it is a Fredholm operator with index zero by [1, Theorem 4.2.1, Corollary 3.2.10]. Now by [9, Corollary 3.11] $T - \gamma I$ is a compact perturbation of an invertible operator, so by Weil's theorem γ is isolated and has finite multiplicity. □

LEMMA 8: *If R is an essentially transitive subalgebra, then R contains an operator T with the following properties: $1 \in \sigma_F(T)$ and if $\gamma \in \sigma(T)\backslash\sigma_F(T)$ then $|\gamma| < \frac{1}{2}$.*

Proof: Let y be a functional in B^*, $|y| = 3$, and let Q be a ball of radius 2 with center in y. Then Q is weak* compact and $\{0\} \notin Q$. The function $h(q) \equiv q$ is obviously weak* continuous, so $h(q) \in WQ$ and $|h(q)| = 5$. Let M and N be subspaces of WQ as in Lemma 6. If N is not dense in M, then there exists a functional $\theta \in M^*$ which is an extreme point of $U_{N^\perp}$. Then by Lemma 6 there exists a point $q_0 \in Q$ and a nonzero element $x \in B^{**}$ s.t. the inequality

$$|(x, A^*h(q_0))| \leq 2|||A|||\ |h|$$

holds for every operator $A \in R$. For $y = h(q_0)/2|h|$ we get inequality (1). This contradiction gives that N is dense in M. Let g be a function in SQ defined by $g(q) \equiv y$. Then there exists an operator-valued function $A(q) \in NR$ s.t. $|A(q)h(q) - g(q)| < \frac{1}{8}$. Thus the function $A(q)$ generates a continuous map $\Psi : Q \to Q$ by the formula $\Psi(q) = A(q)q$. By Tychonoff's fixed point theorem there exists a point $q_0 \in Q$ s.t. $A(q_0)q_0 = q_0$. We let T denote $A(q_0)$. Let V be open set in Q defined by

$$V = \{q \in Q : \|A(q) - T\| < \tfrac{1}{48}\}.$$

Then

$$\begin{aligned}\sup_{q\in V}|Tq - y| &\leq \sup_{q\in V}|(T - A(q))q| + \sup_{q\in V}|(A(q)q) - y| \\ &\leq \tfrac{5}{48} + \tfrac{1}{8} < \tfrac{1}{4}\end{aligned}$$

If $V_1 = \{z : |z| \leq 1, z + q_0 \in V\}$ then V_1 is a weak* open subset of the unit ball U_{B^*} that contains $\{0\}$ and

$$\begin{aligned}\sup_{z\in V_1}|Tz| &= \sup_{z\in V_1}|T(z + q_0) - Tq_0| \\ &\leq \sup_{q\in V}|Tq - Tq_0| \\ &\leq \sup_{q\in V}|Tq - y| + |Tq_0 - y| \\ &< \frac{1}{4} + \frac{1}{4} = \frac{1}{2}\end{aligned}$$

By definition of weak* topology there exists a subspace $L \subset B^*$ of finite codimension s.t. $L \cap U_{B^*} \subset V_1$. We have

$$\sup_{z\in U_L}|Tz| \leq \sup_{z\in V_1}|Tz| < \frac{1}{2},$$

so by Lemma 7 every point $\gamma \in \sigma(T)$, $|\gamma| \geq \frac{1}{2}$ is an isolated point in $\sigma(T)$ of finite multiplicity. □

LEMMA 9: *Let R be a uniformly closed essentially transitive subalgebra $L(B)$, then R contains a nonzero finite-dimensional projection.*

Proof: Let T be an operator as in Lemma 8. Since $\{\gamma : (T-\gamma I)^{-1} \in R\}$ is a component of the resolvent set of T it follows that if $(T-\gamma I)^{-1}$ exists and $|\gamma| \geq \frac{1}{2}$ then $(T-\gamma I)^{-1} \in R$.

By Lemma 8 there exists circle $\alpha \subset C$ s.t. 1 is the only point of $\sigma(T)$ inside α and for every point $\gamma \in \alpha$ $(T-\gamma I)^{-1} \in R$. By the Riesz theorem

$$P = \frac{1}{-2\pi i}\int_\alpha (T-\gamma I)^{-1} d\gamma$$

is a nonzero finite-dimensional projection and $P \in R'$. If $P = P_1^*$ then P_1 is a finite-dimensional projection in R. □

Now to finish the proof of Theorem 1 we need the following well-known fact [8].

LEMMA 10: *If a transitive algebra R contains a nonzero finite-dimensional operator, then R is weakly dense in $L(B)$.*

Essential transitivity implies transitivity, so by combining Lemma 9 and Lemma 10 we finish the proof of Theorem 1. □

Of course Lemma 9 gives the possibility to obtain different results on density. For example one of them is

THEOREM 3: *If H is a Hilbert space and R is a uniformly closed essentially transitive subalgebra of $L(H)$, then R contains all compact operators in $L(H)$.*

Now we'll prove the equivalence of Theorem 1 and Theorem 2 in Hilbert space. We first prove that Theorem 2 implies Theorem 1 if B is Hilbert space.

Assume that Theorem 2 holds. Let $C = \overline{\lim}_\alpha |x_\alpha| \cdot |y_\alpha|$. If $A \in R$ then

$$\begin{aligned} 0 &= \lim_\alpha (Ax_\alpha, y_\alpha) \\ &= \lim_\alpha (A(x_\alpha - x), y_\alpha - y) - (Ax, y) + \lim_\alpha (Ax, y_\alpha) + \lim_\alpha (Ax_\alpha, y) \end{aligned}$$

where

$$\lim_\alpha (Ax, y_\alpha) = \lim_\alpha (Ax_\alpha, y) = (Ax, y).$$

Thus

$$(Ax, y) = \lim_\alpha (A(x - x_\alpha), (y_\alpha - y)).$$

If K is a compact operator, then

$$\begin{aligned}|(Ax,\ y)| &\le \overline{\lim_{\alpha}}|((A-K)(x-x_\alpha),\ (y_\alpha-y))| + \overline{\lim_{\alpha}}|(K(x-x_\alpha),\ (y_\alpha-y))| \\ &\le \|A-K\|4C.\end{aligned}$$

Now let $z = y/4C$. Then

$$|(Ax,\ z)| \le \inf_K \|A-K\| = |||A|||$$

and so Theorem 1 follows from Theorem 2. □

To prove that Theorem 1 implies Theorem 2 we use the following result (by Glimm) [5]; B is assumed to be Hilbert space.

LEMMA 11: *Let $\Psi \in L(B)^*$ and $\Psi \in K(B)^\perp$. Then there is a pair of bounded nets (x_α) and (y_α) s.t.*
(1) $w-\lim_\alpha x_\alpha = 0,\ w-\lim_\alpha y_\alpha = 0$
(2) *For every $A \in L(B), \Psi(A) = \lim_\alpha(Ax_\alpha,\ y_\alpha)$.*

Now let $\Psi_1(A) = (Ax, y)$ where x, y are as in Theorem 1, $A \in R$. Theorem 1 gives that Ψ_1 can be extended to span $(R, K(B))$ by putting $\Psi_1(T) = 0$ for $T \in K(B)$. Now let Ψ be the Hahn–Banach extension of the Ψ_1 to all of $L(B)$. We now use Lemma 1 and get Theorem 2 with $x_\alpha = x + x_\alpha$ and $y_\alpha = y - y_\alpha$.

Finally we mention that in the case of a nonreflexive Banach space, Theorem 1 gives invariant subspace corollaries only in the dual space B^*.

Since Enflo in 1976 [4] showed that there are counter-examples to the invariant subspace problem in general Banach spaces, this may be a sign that the following result can be true: If A is a bounded linear operator in a Banach space, then A^* has a nontrivial invariant subspace.

The known counterexamples do not contradict this conjecture. By Corollary 1, this would be true if the following is true: If A is a bounded linear operator in a Banach space, then there exists an algebra with PS property which contains the operator A.

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